

RELATIVISTIC SPINNING PARTICLE IN A NON-COMMUTATIVE EXTENDED SPACETIME

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Abstract:

The relativistic spinning particle model, proposed in [3,4], is analyzed in a Hamiltonian framework. The spin is simulated by extending the configuration space by introducing a light-like four vector degree of freedom. The model is heavily constrained and constraint analysis, in the Dirac scheme, is both novel and instructive. Our major finding is an associated novel non-commutative structure in the extended space. This is obtained in a particular gauge. The model possesses a large gauge freedom and hence a judicious choice of gauge becomes imperative. The gauge fixed system in reduced phase space simplifies considerably for further study. We have shown that this non-commutative phase space algebra is essential in revealing the spin effects in the particle model through the Lorentz generator and Hamiltonian equations of motion.

1 Introduction:

A Relativistic Spinning Particle Model (RSPM), even in the classical context, is quite hard to construct (see for example [1]). One of the earlier examples in physical $3 + 1$ -dimensions, (that we have extensively used in our work [2] in $2 + 1$ -dimensional anyons), is that of Hanson and Regge [3]. The model [3] had a drawback (indeed depending on one's point of view) in that the rest mass and spin parameters are related in a complicated way, *i.e.*

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the mass and spin are *not* independent. (This problem was resolved later by one of us [2] in $2 + 1$ -dimensions giving rise to the spinning particle model for anyon. For alternative formulations of relativistic spinning particle models see [4].) In this perspective it is very significant that another RSPM was proposed some years ago by Kuzenko, Lyakhovich and Segal [5] and rediscovered independently by Staruszkiewicz [6]. The scheme of [5, 6] is based on Wigner’s idea [7] that quantum mechanical systems ought to be classified in terms of unitary irreducible representations of Poincare group. In a classical setup, this can be interpreted [5, 6] as having Casimir invariants that are restricted to fixed numerical values (*i.e.* they have to be parameters of the model rather than constants of motion).

The essential idea behind the present approach [5, 6] is to generate the particle spin through the introduction of additional degrees of freedom, in the form of a light-like four vector $k_\mu(\tau)$. This dynamical variable has the same status as the particle position $x_\mu(\tau)$. Quite clearly this is an example of enlarging the configuration space to a bosonic “superspace”. In fact fermionic superspace construction to describe spinning particles was studied much earlier [8].

In the present context of spinning particle it is natural to choose the two Casimirs as

$$P^\mu P_\mu = m^2 ; \quad W^\mu W_\mu = -(l^2 m^2)/4 ; \quad W_\mu = -(1/2)\epsilon_{\mu\nu\rho\sigma} M^{\nu\rho} P^\sigma, \quad (1)$$

with P_μ and W_μ being the canonical momentum and Pauli-Lubanski pseudovector respectively. m and l are the mass and spin parameters of the theory but infact l has the dimension of length. It was established in [5, 6] and later in more general context by Kassandrov et.al [9] that (1) can lead to an unambiguous form of action (we will come to it presently) from which P_μ and W_μ can be computed and the identities in (1) checked. Subsequently the authors of [9] have carried out a Lagrangian analysis of this particular RSPM and have come up with interesting observations.

In this perspective our aim is to make an exhaustive Hamiltonian analysis of the RSPM proposed in [5, 6] in a gauge fixed framework. The NC spacetime appears in our reduced phase space. It should be mentioned that in [5] a Hamiltonian analysis was performed in

the extended phase space and hence the NC nature of the spacetime did not show up. The reasons of our study are primarily twofold but connected.

Reason (I): The beauty of this model [5, 6] (along with the previous RSPMs [3, 2], for other models having similar features see *eg.* [10]) is that all of them possess a non-trivial symplectic structure which induces Non-Commutative (NC) spacetime (or in the more general context NC phase space) and this noncommutativity induces the richer dynamics that can incorporate the spin. *Presence of other numerical parameters, besides the mass, in a free particle Lagrangian is a signal for this where eventually the other parameter plays the role of NC parameter.* Different forms of NC algebra have appeared in the above mentioned RSPMs [3, 2, 10]. Hence from our previous experience we could make an educated guess that the RSPM of [5, 6] should also have an NC phase space [11], which, incidentally, will be completely new since the peculiar symplectic structure of this form has not been explored before. However, as it turned out, the constraint analysis of this model [5, 6], in the Hamiltonian formulation of Dirac [12], is quite non-trivial and interesting. As we will show later, the model possesses a large amount of gauge freedom and hence a judicious gauge choice becomes necessary. Our particular system of gauges is dictated by the natural requirement that we end up with a (albeit NC) phase space where the Lorentz generators can be defined unambiguously ensuring that the vectors transform in a covariant manner. Furthermore, the gauge fixed Hamiltonian reduces to that of a canonical relativistic spinless particle, (as in the case of [3, 2]), and spin effects in the dynamics are generated through the NC Dirac bracket algebra. We will elaborate on this when we get down to the actual computation.

Reason (II): Kassandrov et.al. [9] made an intriguing comment that for appropriate choice of the parameters, in particular identifying l as the Compton length $i = \hbar/(mc)$ the particle spin can be $\hbar/2$. Now, it will be really worthwhile to attempt a quantization of the model and that will require a Hamiltonian formulation, symplectic structures, etc.. In this sense, the present work is a stepping stone for quantizing this spinning particle.

The paper is organized as follows: In **Section 2** we will introduce the Lagrangian of

RSPM [5, 6] and reveal the constraints in the theory. It will be seen that we need to introduce auxiliary fields in order to reduce the model to a form that is amenable to Hamiltonian constraint analysis. For completeness a brief outline of the Dirac constraint analysis [12] will be provided. **Section 3** will be devoted to the study of the symplectic structure in a particular set of gauges so that in the reduced phase space the NC phase space algebra will be revealed. As we have mentioned, the problem of gauge fixing is quite non-trivial. In **Section 4** the dynamics of the model will be studied in this particular gauge where the NC phase space algebra will be used explicitly. Throughout we will show the correspondence between our results (obtained in the Hamiltonian framework) to that of the same model of [9] (computed in the Lagrangian framework). We will conclude the paper in **Section 5** with future prospects.

2 Relativistic Spinning Particle Model:

The action for the RSPM, as stated in [6] is,

$$S = \int L d\tau = -m \int d\tau \sqrt{\dot{x}^2} \sqrt{1 + l \sqrt{\frac{(-\dot{k}^2)}{(k\dot{x})^2}}} + \int d\tau \lambda k^2. \quad (2)$$

As we pointed out in the Introduction, the action L has two parameters m , the particle mass, and l the spin parameter, having dimension of length in our system of units. As it happens in these type of models [3, 2, 10] l will play the role of the NC parameter; k_μ is a lightlike vector, $k^\mu k_\mu = k^2 = 0$. We use a shorthand notation $(ab) = a^\mu b_\mu$ and the metric $g_{\mu\nu} = \text{diag}(1, -1, -1, -1)$.

The Lagrangian (2) can be considered as a nontrivial extension of the Nambu-Goto form of spinless particle, to which (2) reduces to for $l = 0$, since the λk^2 term is non-dynamical and gets decoupled from the dynamical term $-m\sqrt{\dot{x}^2}$. One can directly define the conjugate momenta and angular momentum [9]

$$P_\mu = \frac{\partial L}{\partial \dot{x}^\mu} ; \quad Q_\mu = \frac{\partial L}{\partial \dot{k}^\mu} ; \quad M_{\mu\nu} = x_\mu P_\nu - x_\nu P_\mu + k_\mu Q_\nu - k_\nu Q_\mu$$

and check explicitly that the Casimir relations (1) are satisfied.

However for Hamiltonian analysis, this form of the model, with its involved time derivative structure and presence of nested square roots, is simply not suitable because one will immediately run into trouble in trying to express the velocities \dot{x}_μ and \dot{k}_μ in terms of the momenta P_μ and Q_μ . For this we need to remove the derivatives (recall the analogous analysis for the spinless particle) which can be done at the cost of introducing auxiliary Degrees Of Freedom (DOF). Simply stated we will work in the Polyakov form since classically Nambu-Goto and Polyakov forms are equivalent (on shell). Furthermore we will exploit a trick, introduced by Lukierski et.al. [13], where one introduces auxiliary variables, identified with time derivatives of physical variables, to reduce or simplify the overall time derivative structure of the Lagrangian. Indeed this prescription does not lead to the simplified form in a unique way so we will consider a form that is convenient for our purpose. (Of course, for consistency of the scheme of [13], different explicit forms, obtained due to this ambiguity, are physically equivalent.)

Hence instead of (2) let us consider the classically equivalent form,

$$L = p_\mu(\dot{x}^\mu - y^\mu) - \frac{y^2}{2e} - \frac{ly^2}{2e} \frac{\sqrt{-\dot{k}^2}}{(ky)} - \frac{em^2}{2} + \lambda k^2. \quad (3)$$

Following [13] we have replaced \dot{x}^μ by y^μ and this identification is enforced by the first term with p_μ acting as a Lagrange multiplier. From (3) the original form (2) can be recovered by integrating out e . The advantage of extending the space of variables is that now the Lagrangian is tractable for constraint analysis.

We now have on our hand a first order form of Lagrangian (3) where all the degrees of freedom are to be treated as independent variables. The long list of conjugate momenta reads:

$$p_\mu^x = \frac{\partial L}{\partial \dot{x}^\mu} = p_\mu, \quad p_\mu^p = 0, \quad p_\mu^y = 0, \quad p^e = 0, \quad p^\lambda = 0, \quad p_\mu^k = \frac{ly^2}{2e} \frac{\dot{k}_\mu}{(ky)\sqrt{-\dot{k}^2}}. \quad (4)$$

The canonical Hamiltonian H follows,

$$H = p_\mu^x \dot{x}^\mu + p_\mu^p \dot{p}^\mu + p_\mu^y \dot{y}^\mu + p_\mu^k \dot{k}^\mu + p^e \dot{e} + p^\lambda \dot{\lambda} - L = (py) + \frac{y^2}{2e} + \frac{em^2}{2} - \lambda k^2. \quad (5)$$

Notice that in (4) only \dot{k}_μ and no other velocities appear. Hence majority of the velocities remain undetermined meaning that there are constraints in the system. This is not surprising since we have converted the L of (2) to a first order one in (3).

Before proceeding further we make a quick digression to discuss the relevant points of Dirac's Hamiltonian analysis of constrained systems [12].

Dirac's constraint analysis: In Hamiltonian formulation, any relation between dynamical variables, *independent of velocities*, is considered as a constraint. Constraints can appear directly in defining the conjugate momenta, (as in (4) in the present case). New constraints can also be generated from demanding time persistence of the first set of constraints.

In the full set of constraints, the ones that commute with all others (in the sense of Poisson brackets) are termed as First Class Constraints (FCC). Rest of the non-commuting constraints are termed as Second Class Constraints (SCC). The FCCs and SCCs have to be treated in essentially different ways, especially if the system is being quantized.

Presence of constraints signifies a redundancy in the number of DOF involved. FCCs induce presence of local gauge invariance. FCCs can be treated in two ways. One can keep all the DOFs and impose the FCCs by restricting the set of physical states to those satisfying $(FCC) | state \rangle = 0$. On the other hand, one can choose additional constraints, (one each for one FCC), known as gauge fixing conditions so that, these together with the FCCs turn in to an SCC set.

For SCCs, a similar relation as above, $(SCC) | state \rangle = 0$ can not be implemented consistently and one needs to modify the symplectic structure. Poisson brackets have to be replaced by Dirac bracket. Between two generic variables A and B it is defined as,

$$\{A, B\}_{DB} = \{A, B\} - \{A, (SCC)_i\} \{ (SCC)_i, (SCC)_j \}^{-1} \{ (SCC)_j, B \}, \quad (6)$$

where $(SCC)_i$ is a set of SCC and $\{ (SCC)_i, (SCC)_j \}$ is the constraint matrix. Upon quantization, the Dirac brackets are elevated to quantum commutators. For SCCs this matrix is invertible. After exploiting the SCCs strongly as operator relations and working with Dirac brackets one can work in a reduced phase space, (as we will discuss in **section 3**), where

SCCs are used to eliminate some DOF in favor of others.

The relevance of constraint analysis and Dirac brackets in the context of NC geometry lies in the fact that the constraints present in the model induce Dirac brackets that determine the NC phase space algebra.

It is time now to consider our model. The first batch of constraints, (known as Primary constraints [12]), appear directly from (4):

$$\begin{aligned}\psi_\mu^{(1)} &\equiv p_\mu^x - p_\mu \quad , \quad \psi_\mu^{(2)} \equiv p_\mu^p \quad , \quad \psi_\mu^{(3)} \equiv p_\mu^y \quad , \quad \psi^{(4)} \equiv p^e \quad , \\ \psi^{(5)} &\equiv p^\lambda \quad , \quad \psi^{(9)} \equiv (p^k)^2 + \frac{l^2(y^2)^2}{4e^2(ky)^2}.\end{aligned}\tag{7}$$

The last one, $\psi^{(9)}$ follows from “squaring” the last relation in (4) involving p_μ^k . (This is similar to deriving the mass-shell constraint from the Nambu-Goto form of free spinless particle Lagrangian.) Invariance under time translation yields the remaining Secondary constraints [12],

$$\begin{aligned}\dot{\psi}_\mu^{(3)} = \{\psi_\mu^{(3)}, H\} &\equiv \psi_\mu^{(6)} = p_\mu + \frac{y_\mu}{e} \quad , \quad \dot{\psi}^{(4)} + (p^\mu - \frac{y^\mu}{e})\psi_\mu^{(6)} \equiv \psi^{(7)} = (p^x)^2 - m^2 \quad , \\ \dot{\psi}^{(5)} &\equiv \psi^{(8)} = k^2 \quad , \quad \dot{\psi}^{(9)} \equiv \psi^{(10)} = (kp^k).\end{aligned}\tag{8}$$

We use the convention for Poisson bracket and the metric as,

$$\{x_\mu, p_\nu^x\} = -g_{\mu\nu} \quad ; \quad \{k_\mu, p_\nu^k\} = -g_{\mu\nu} \quad ; \quad g_{\mu\nu} = \text{diag}(1, -1, -1, -1).$$

Note that time derivatives of $\psi_\mu^{(1)}$ and $\psi_\mu^{(2)}$ are not considered since this is a trivial set of SCC which is solved at once and p_μ is replaced by p_μ^x . This will not change the algebra the remaining DOF. Again $\psi^{(5)}$ is a trivial FCC which we remove by fixing a gauge $\lambda = 1$.

After trimming down the set of constraints to the non-trivial ones we must make the essential classification of FCCs and SCCs among them. For convenience we simplify and rename the constraints as $\eta^{(1)}$, $\eta^{(2)}$, $\eta^{(3)}$ for $\psi^{(7)}$, $\psi^{(8)}$, $\psi^{(10)}$ and $\phi_\mu^{(1)}$, $\phi_\mu^{(2)}$, $\phi^{(3)}$, $\phi^{(4)}$ for $\psi_\mu^{(3)}$, $\psi_\mu^{(6)}$, $\psi^{(4)}$, $\psi^{(9)}$ so that we have the following sets:

$$\eta^{(1)} \equiv (p^x)^2 - m^2 \quad , \quad \eta^{(2)} \equiv k^2 \quad , \quad \eta^{(3)} \equiv (kp^k),\tag{9}$$

$$\phi_\mu^{(1)} \equiv p_\mu^y, \quad \phi_\mu^{(2)} \equiv p_\mu^x + \frac{y_\mu}{e}, \quad \phi^{(3)} \equiv p^e, \quad \phi^{(4)} \equiv (p^k)^2 + \frac{m^2 l^2 y^2}{4(ky)^2}. \quad (10)$$

One can directly check that the set η in (9) are FCC that is they satisfy a closed algebra with all the constraints η, ϕ . From the ϕ set in (10) one can construct two combinations,

$$\eta^{(4)} \equiv e\phi^{(3)} + (y\phi^{(1)}); \quad \eta^{(5)} \equiv \phi^{(4)} + \frac{em^2 l^2 y^2}{2(ky)^3}(k\phi^{(2)}) - \frac{em^2 l^2}{2(ky)^2}(y\phi^{(2)}), \quad (11)$$

that are also FCC. Below we provide the non-abelian FCC algebra of the set η in ((9),(11)):

$$\begin{aligned} \{\eta^{(2)}, \eta^{(3)}\} &= -2\eta^{(2)}, \quad \{\eta^{(2)}, \eta^{(5)}\} = -4\eta^{(3)}, \\ \{\eta^{(3)}, \eta^{(5)}\} &= -2\eta^{(5)}, \quad \{\eta^{(4)}, \phi_\mu^{(1)}\} = -\phi_\mu^{(1)}. \end{aligned} \quad (12)$$

The remaining SCC set satisfies,

$$\{\phi_\mu^{(1)}, \phi_\nu^{(2)}\} = \frac{1}{e}g_{\mu\nu}. \quad (13)$$

We can solve this SCC system and the subsequent analysis will remain unaffected. We find that the canonical Hamiltonian $H = (p^x y) + \frac{y^2}{2e} + \frac{em^2}{2}$ vanishes and hence the dynamics will be governed by the FCCs only as is the case of reparametrization invariant theories. Hence the extended Hamiltonian, in $e = 1$ gauge, is,

$$H = \lambda_1((p^x)^2 - m^2) + \lambda_2 k^2 + \lambda_3(kp^k) + \lambda_4 \left((p^k)^2 + \frac{m^4 l^2}{4(kp^x)^2} \right), \quad (14)$$

where λ_i are undetermined multipliers. It is straightforward to check that the constraint with λ_4 , in conjunction with the other constraints, reproduces the Pauli-Lubanski condition in (1).

As we have mentioned in the discussion on Dirac procedure [12] there are two ways of tackling a system with FCC. In the gauge invariant formulation no gauge is fixed and one works with all the DOF. We study this scheme in this section. In the next section, **section 3**, we will analyze the gauge fixed version, which, incidentally is our main concern.

We now need to fix the multiplier λ_i . For this let us follow Gitman and Tyutin [14]. The idea is to determine λ_i by comparing the expressions for velocities obtained from (14),

$$\dot{x}_\mu = \{x_\mu, H\} = -2\lambda_1 p_\mu^x + \lambda_4 \frac{m^4 l^2 k_\mu}{2(kp^x)^3}, \quad (15)$$

and from the original action (2),

$$p_\mu^x = (\partial L)/(\partial \dot{x}^\mu). \quad (16)$$

Solving the above relations we find,

$$\lambda_1 = (\sqrt{\dot{x}^2}) / \left(2m \sqrt{1 + l \sqrt{\frac{(-\dot{k}^2)}{(k\dot{x})^2}}} \right) \quad ; \quad \lambda_4 = -l(k\dot{x}) \sqrt{(-\dot{k}^2) \left(1 + l \sqrt{\frac{(-\dot{k}^2)}{(k\dot{x})^2}} \right)}. \quad (17)$$

Similarly for the other DOF k_μ we compare,

$$\dot{k}_\mu = \{k_\mu, H\} = -\lambda_3 k_\mu - 2\lambda_4 p_\mu^k, \quad (18)$$

with

$$p_\mu^k = (\partial L)/(\partial \dot{k}^\mu) \quad (19)$$

to obtain

$$\lambda_3 = 0. \quad (20)$$

Hence λ_2 remains arbitrary. In determining the multipliers we have used the Dirac brackets obtained from the SCC $\phi_\mu^{(1)}, \phi_\mu^{(2)}$ which, however, do not alter the brackets needed to compute ((15),(18)). This is because the set of SCC ϕ can modify only brackets of the generic form $\{y_\mu, \}$ but the H in (14) does not contain y_μ ³.

As a demonstration of the correctness of our constraint analysis and the (Dirac) classification of constraints, one can go back to ((15),(18)), substitute λ_i explicitly to check that the equations of motion for $\dot{x}_\mu, \dot{k}_\mu, \dot{p}_\mu^x, \dot{p}_\mu^k$ are identical with those of [9].

3 New Non-Commutative Spacetime:

In this section our aim is to construct a minimal or *reduced* NC phase space by exploiting the constraint relations to eliminate some of the DOF. This is possible if one introduces gauge fixing constraints for the FCCs and compute Dirac brackets for the complete SCC

³See the comment below (10).

system, consisting of FCCs $\eta^{(1)} - \eta^{(5)}$, gauge fixing constraints to be given and the original SCCs $\phi_\mu^{(1)}, \phi_\mu^{(2)}$. One can perform a quick counting to check how many DOFs survive a full gauge fixing. In the first order Lagrangian (3) we have the phase space variables $x_\mu, k_\mu, y_\mu, e, p_\mu, p_\mu^x, p_\mu^k, p_\mu^y, p^e$ meaning 26 variables in total. Using 5 FCC and 8 SCC we can remove in all 18 variables so that we are left with 8 DOF in phase space that can have dynamics. We might choose these to be x_μ, p_μ^x . This means that, in principle, we can finally have a Hamiltonian consisting of x_μ, p_μ^x only and Dirac brackets for x_μ, p_μ^x the latter being the NC phase space for the RSPM of [6] that we have been advertising.

Indeed, the gauge fixing process is not unique and one fixes gauge conditions according to the specific model in question as well as the particular goal one has in mind. Our aim is to project this RSPM as an extension to the spinless relativistic particle. For the latter one has $L \sim m\sqrt{\dot{x}^2}$ leading to a single FCC $\eta \sim (p^x)^2 - m^2$. From here the gauge fixing for reparametrization invariance (to fix the time variable) and subsequent quantization can follow the analysis given in [14]. This means that in the present case, out of the 5 FCCs $\eta_1 - \eta_5$ we will fix all but the first one, $\eta^{(1)} = (p^x)^2 - m^2$. This will make formally the Hamiltonian of the RSPM same as the spinless one and all the complexity will be present in the NC phase space algebra. Of course these NC Dirac brackets will reduce to the canonical Poisson brackets for $l = 0$. Hence l is to interpreted as the NC parameter.

Furthermore, this particular gauge system that we have introduced has another important property: It allows a simple structure of Lorentz generator even in the very complicated NC manifold and all the dynamical phase space variables transform covariantly once the NC algebra is used.

In the present model a non-trivial technical problem arises in the explicit gauge fixing process since our aim is to keep η_1 as a FCC so that one can fix time in the conventional way [14]. This means that the gauge conditions have to be chosen such that they commute with η_1 . The gauge fixing and the resulting Dirac bracket computation is quite involved. In the **Appendix** we give an outline of the gauge fixing procedure and a particular gauge fixing

for which we now provide the Dirac brackets. Note that we will provide the NC algebra in terms of the variables x_μ , p_μ^x , k_μ , p_μ^k although we have fixed all the FCC except η_1 . This means that in principle we could have given only the algebra consisting of x_μ , p_μ^x (without k_μ , p_μ^k) but that will possibly be a very complicated algebra.

Below we provide the NC phase algebra (or Dirac brackets) for RSPM [6] in a specific set of gauges (see **appendix** for details) is. First comes the x_μ, p_μ^x sector:

$$\begin{aligned}\{x_\mu, x_\nu\}_{DB} &= \frac{m^2 l^2 (x_\mu k_\nu - k_\mu x_\nu)}{4c^2 (xp^k)} ; \quad \{p_\mu^x, p_\nu^x\}_{DB} = 0 , \\ \{x_\mu, p_\nu^x\}_{DB} &= -g_{\mu\nu} - \frac{m^2 l^2 (m^2 k_\mu k_\nu - c k_\mu p_\nu^x)}{4c^3 (xp^k)}\end{aligned}\quad (21)$$

Then we give the k_μ, p_μ^k Dirac brackets,

$$\begin{aligned}\{k_\mu, k_\nu\}_{DB} &= \frac{c(k_\nu p_\mu^k - k_\mu p_\nu^k)}{m^2 (xp^k)} , \\ \{p_\mu^k, p_\nu^k\}_{DB} &= -\frac{m^4 l^2 (x_\mu p_\nu^x - x_\nu p_\mu^x)}{4c^3 (xp^k)} + \left(\frac{1}{c} - \frac{m^2 l^2}{4c^2 (xp^k)} \right) (p_\mu^x p_\nu^k - p_\mu^k p_\nu^x) \\ &\quad + \frac{m^4 l^2}{4c^4 (xp^k)} (m^2 (k_\mu x_\nu - x_\mu k_\nu) - (xp^x)(k_\mu p_\nu^x - k_\nu p_\mu^x)), \\ \{k_\mu, p_\nu^k\}_{DB} &= -g_{\mu\nu} + \frac{m^2 p_\mu^k x_\nu - (xp^x) p_\mu^k p_\nu^x}{m^2 (xp^k)} + \frac{1}{c} (k_\mu p_\nu^x - k_\nu p_\mu^x) - \frac{m^2 l^2 k_\mu p_\nu^x}{4c^2 (xp^k)} - \frac{c p_\mu^k p_\nu^k}{m^2 (xp^k)}.\end{aligned}\quad (22)$$

Finally we have the mixed brackets,

$$\begin{aligned}\{x_\mu, k_\nu\}_{DB} &= -\frac{((xp^x)k_\mu + cx_\mu)p_\nu^k}{m^2 (xp^k)} - \left(-\frac{1}{c} + \frac{m^2 l^2}{4c^2 (xp^k)} \right) k_\mu k_\nu , \\ \{x_\mu, p_\nu^k\}_{DB} &= -\frac{m^2 l^2 (m^2 k_\mu x_\nu + cx_\mu p_\nu^x)}{4c^3 (xp^k)} - \frac{1}{c} (k_\nu p_\mu^k + k_\mu p_\nu^k) + \frac{m^2 l^2 k_\mu p_\nu^k}{4c^2 (xp^k)} - \frac{m^4 l^2 ((xp^x)k_\mu k_\nu + cx_\mu k_\nu)}{4c^4 (xp^k)}, \\ \{k_\mu, p_\nu^x\}_{DB} &= \frac{m^2 p_\mu^k k_\nu - c p_\mu^k p_\nu^x}{m^2 (xp^k)} \\ \{p_\mu^x, p_\nu^k\}_{DB} &= -\frac{m^2 l^2 (m^2 k_\mu p_\nu^x - c p_\mu^x p_\nu^x)}{4c^3 (xp^k)} + \frac{m^4 l^2 k_\nu p_\mu^x}{4c^3 (xp^k)} - \frac{m^6 l^2 k_\mu k_\nu}{4c^4 (xp^k)}\end{aligned}\quad (23)$$

In the above NC algebra $c \neq 0$ is a numerical parameter that appears from gauge fixing (see **Appendix**).

One point regarding the above NC algebra (3,3,3) should be mentioned. For $l = 0$ the x_μ, p_μ^x sector reduces to the canonical one but rest of the brackets, although simplifies considerably, still remain non-trivial.

The above NC algebra can be put to a non-trivial use: generating the Lorentz algebra from the covariant angular momentum operator $M_{\mu\nu}$. It is important to note that using $((3),(3),(3))$, algebra among $x_\mu p_\nu^x - x_\nu p_\mu^x$, (which is the angular momentum for the spinless particle), does not close. But the correct Lorentz algebra,

$$\{M_{\mu\nu}, M_{\alpha\beta}\}_{DB} = g_{\mu\beta}M_{\nu\alpha} + g_{\nu\alpha}M_{\mu\beta} + g_{\mu\alpha}M_{\beta\nu} + g_{\nu\beta}M_{\alpha\mu}, \quad (24)$$

is recovered only when the angular momentum has a spin contribution. The correct generator is⁴,

$$M_{\mu\nu} = x_\mu p_\nu^x - x_\nu p_\mu^x + k_\mu p_\nu^k - k_\nu p_\mu^k. \quad (25)$$

The demonstration of the above is straightforward but quite tedious and involves a number of non-trivial cancellations between terms coming from different sectors. The structure of $M_{\mu\nu}$ is quite elegant when contrasted to the more involved form given in [9].

As we have already advertised it is straightforward to explicitly check, using the NC Dirac brackets (3,3,3), that this $M_{\mu\nu}$ reproduces the usual Lorentz transformation for all the physical degrees of freedom:

$$\begin{aligned} \{M_{\mu\nu}, x_\rho\}_{DB} &= g_{\nu\rho}x_\mu - g_{\mu\rho}x_\nu \quad , \quad \{M_{\mu\nu}, p_\rho^x\}_{DB} = g_{\nu\rho}p_\mu^x - g_{\mu\rho}p_\nu^x \quad , \\ \{M_{\mu\nu}, k_\rho\}_{DB} &= g_{\nu\rho}k_\mu - g_{\mu\rho}k_\nu \quad , \quad \{M_{\mu\nu}, p_\rho^k\}_{DB} = g_{\nu\rho}p_\mu^k - g_{\mu\rho}p_\nu^k. \end{aligned} \quad (26)$$

4 Spinning Particle Dynamics:

We have entered the final stage of our analysis of the RSPM of [6] where we derive the Hamiltonian dynamics by fixing the time in the canonical gauge (for details see [14]). The

⁴The generator $M_{\mu\nu}$ is gauge invariant which can be seen easily because it commutes with all the FCC listed in the expression for the Hamiltonian in (14), where one is free to use canonical Poisson brackets. It is a conserved quantity as well.

essential point is that one rewrites the remaining FCC $\eta_1 = (p^x)^2 - m^2$ as,

$$\eta_1 = |p_0| - \sqrt{p_i^x p_i^x + m^2}, \quad (27)$$

and fix the gauge $x'_0 = 0$. This amounts to a gauge fixing of the form $x'_0 \sim x_0 - \tau = 0$ with $(x_i)' = x_i$, $(p_\mu)' = p_\mu$. This requires x_0 to be a c -number parameter (time) and this formal manipulation is convenient because one can then directly apply the Dirac analysis with the time-independent constraint $\sim x'_0 = 0$ (we follow [14]). In the spinless case, with this gauge choice one recovers the dynamics of physical variables $\dot{x}_i = p_i / \sqrt{p_i^x p_i^x + m^2}$ from the physical reduced space Hamiltonian $H = \sqrt{p_i^x p_i^x + m^2}$ using Dirac brackets following from the SCCs $x'_0, |p_0| - \sqrt{p_i^x p_i^x + m^2}$.

In the present case also we need to carry out the above procedure keeping in mind the last iteration of Dirac brackets $((3),(3),(3))$. Specifically we consider the FCC and gauge fixing,

$$\eta_1 = |p'_0| - \sqrt{(p^x)'_i (p^x)'_i + m^2}, \quad x'_0 = 0, \quad (28)$$

(where it is simply a renaming for $(x_i)' = x_i$, $(p_\mu)' = p_\mu$), and compute the last stage Dirac brackets using the penultimate stage Dirac brackets $(3,3,3)$. The final NC algebra for the physical x_i, p_i^x sector is,

$$\begin{aligned} \{x_i, x_j\}_{DB} &= \frac{l^2 m^2}{4c^2 (xp^k)} \left(x_i \left(k_j + \frac{k_0 p_j}{\sqrt{\vec{p}^2 + m^2}} \right) - x_j \left(k_i + \frac{k_0 p_i}{\sqrt{\vec{p}^2 + m^2}} \right) \right), \\ \{x_i, p_j\}_{DB} &= -g_{ij} - \frac{l^2 m^2}{4c^2 (xp^k)} \left(k_i + \frac{k_0 p_i}{\sqrt{\vec{p}^2 + m^2}} \right) (m^2 k_j - c p_j); \quad \{p_i, p_j\} = 0. \end{aligned} \quad (29)$$

Note that the final Dirac brackets in the k_μ, p_μ^k, p_μ^x sector remains *unchanged* and is same as (3). This occurs due to $\{\eta_1, k_\mu\} = 0$; $\{\eta_1, p_\mu^k\} = 0$; $\{\eta_1, p_\mu^x\} = 0$ using (3) and this calculation is quite amusing. However there will again be changes in the mixed x_μ, k_μ and x_μ, p_μ^k sectors which we have not shown here.

Finally we recover the cherished forms of the equations of motion for the free RSPM [6, 9]:

$$\dot{x}_i = -\frac{p_i}{\sqrt{\vec{p}^2 - m^2}} - \frac{l^2 m^2 (m^2 k_0 - c \sqrt{\vec{p}^2 - m^2})}{4c^3 (xp^k)} \left(k_i + \frac{k_0}{\sqrt{\vec{p}^2 - m^2}} p_i \right), \quad (30)$$

$$\dot{p}_i = 0. \quad (31)$$

For consistency one can check that the same result is obtained either from $\dot{x}_i = \{x_i, p_0\}_{DB}$ or from $\dot{x}_i = \{x_i, \sqrt{\vec{p}^2 - m^2}\}_{DB}$. The dynamics of k_μ can be obtained straight from (3),

$$\dot{k}_\mu = \frac{(m^2 k_0 - c\sqrt{\vec{p}^2 - m^2})}{m^2(xp^k)} p_\mu^k. \quad (32)$$

Another interesting finding is the presence of a conserved pseudovector,

$$G_i = \epsilon_{ijl}(m^2 k_j - cp_j^x) p_l^k ; \quad \dot{G}_i = 0. \quad (33)$$

The equations of motion and the conserved vector G_i are the Hamiltonian analogues of the the Lagrangian equations of motion obtained in [9].

5 Concluding Remarks and Future Prospects

The relativistic spinning particle model, proposed in [5, 6] has several interesting features that need further careful investigation. The main interest lies in the fact the model is a classical one but nevertheless has quantum particle like property as regards to its spin. The spin appears as a Casimir and hence possess a universal character. From a purely mathematical point of view this extension of the standard spinless particle model is quite novel and has origin in group theoretic ideas. Many interesting behaviors of the particle have been revealed in the Lagrangian analysis conducted by Kassandrov et.al. [9].

In the present work we have carried a thorough Hamiltonian analysis of the model. The principal result of our work is the emergence of a new form of non-commutativity in an (extended) spacetime. Using this non-commutative phase space approach we have corroborated our results for the particle dynamics with those of [9]. Furthermore, even if it is treated only as a problem of interest in constrained dynamical system, the Hamiltonian Dirac constraint analysis possesses several subtle and intricate features.

There are a number of avenues open for further study on this model in our proposed Hamiltonian framework. The non-commutative algebra needs to be analyzed carefully since there might be combinations of the fundamental degrees of freedom $(x_\mu, p_\mu^x, k_\mu, p_\mu^k)$ that can lead to a simpler non-commutative symplectic structure which will help to understand the dynamics better. The other (hopefully straightforward) task is to include external interactions in the model along the lines of [9].

Finally, Hamiltonian formulation is the starting point from which one can attempt a proper quantization of the model. In the present case, the way we have formulated the problem, the Hamiltonian has reduced to that of a simple free spinless particle and all the spin related complexity resides in the non-commutative phase space algebra. A similar thing appears in the previous spinning particle models [3, 2] as well. Hence a non-trivial job is to find, at least to lowest order in non-commutativity, a proper representation of the classical variables as quantum operators. Once again, the first step towards this realization lies in deriving a (Darboux like) map from the non-commutative degrees of freedom to canonical degrees of freedom, in the classical setup. This will probably lead to a more complicated Hamiltonian, with spin effects included, but a simpler canonical phase space algebra, so that a perturbative Schrodinger analysis may be performed in a non-relativistic approximation.

Appendix:

We outline computation of the Dirac brackets (3,3,3) resulting in the NC phase space. For a specific system with a large number of SCCs the Dirac brackets can be evaluated iteratively that is one can start by, say, any two SCCs, compute their Dirac bracket, then use this as the starting bracket to compute Dirac brackets for rest of the SCCs. Since in the present case we have three FCCs so to calculate Dirac brackets for a system of six SCCs is difficult. Hence we start with $\eta^{(5)}$ (with $\phi_\mu^{(2)}$ removed) and fix a gauge condition

$$\eta_g^{(5)} \equiv \frac{1}{2} \left((xk) - \frac{(xp^x)(kp^x)}{m^2} \right) = 0. \quad (34)$$

The constraint matrix for the SCC pair $\eta^{(5)}, \eta_g^{(5)}$ and its inverse are given by

$$\{\eta^{(5)}, \eta_g^{(5)}\} = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \quad \{\eta^{(5)}, \eta_g^{(5)}\}^{-1} = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \quad (35)$$

where

$$a = (xp^k) - \frac{(p^k p^x)(xp^x)}{m^2}; \quad b = \frac{m^2}{(p^k p^x)(xp^x) - m^2(xp^k)}.$$

This gives rise to Dirac brackets which are not shown. Then we fix gauges

$$\eta_g^{(3)} \equiv (kp^x) - c = 0; \quad \eta_g^{(2)} \equiv (p^k p^x) = 0, \quad (36)$$

for the remaining two FCCs and again calculate Dirac brackets using the previously obtained Dirac brackets as the starting phase space algebra. In (36) c is a non-zero numerical parameter. It should be mentioned that the specific gauge choices we have made are dictated by our target of keeping the FCC nature of $\eta^{(1)}$ intact. Indeed, one is free to make other (allowed) gauge choices. The SCC constraint matrix consisting of brackets among $\eta_3, \eta_g^{(3)}, \eta_2, \eta_g^{(2)}$ is now 4-dimensional. Below the constraint matrix $[matrix]_{ij}$ is written with the constraints arranged in order with $[matrix]_{11} = \{\eta_3, \eta_3\}$.

$$[matrix]_{ij} = \begin{pmatrix} 0 & a_1 & 0 & 0 \\ -a_1 & 0 & 0 & -a_2 \\ 0 & 0 & 0 & -2a_1 \\ 0 & a_2 & 2a_1 & 0 \end{pmatrix}, \quad [matrix]_{ij}^{-1} = \begin{pmatrix} 0 & -b_1 & b_2 & 0 \\ b_1 & 0 & 0 & 0 \\ -b_2 & 0 & 0 & -b_1/2 \\ 0 & 0 & b_1/2 & 0 \end{pmatrix} \quad (37)$$

where

$$a_1 = (kp^x), \quad a_2 = m^2 + \frac{m^6 l^2}{4(kp^x)((p^k p^x)(xp^x) - m^2(xp^k))}$$

$$b_1 = 1/(kp^x), \quad b_2 = \left(m^2 + \frac{m^6 l^2}{4(kp^x)((p^k p^x)(xp^x) - m^2(xp^k))} \right) / 2(kp^x)^2.$$

From here the Dirac brackets (3,3,3) are computed.

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